TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 353, Number 12, Pages 4899–4924
S 0002-9947(01)02821-5
Article electronically published on July 12, 2001

AUTOMORPHISMS OF THE LATTICE OF Π^0_1 CLASSES; PERFECT THIN CLASSES AND ANC DEGREES

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ABSTRACT. Π^0_1 classes are important to the logical analysis of many parts of mathematics. The Π^0_1 classes form a lattice. As with the lattice of computably enumerable sets, it is natural to explore the relationship between this lattice and the Turing degrees. We focus on an analog of maximality, or more precisely, hyperhypersimplicity, namely the notion of a thin class. We prove a number of results relating automorphisms, invariance, and thin classes. Our main results are an analog of Martin's work on hyperhypersimple sets and high degrees, using thin classes and anc degrees, and an analog of Soare's work demonstrating that maximal sets form an orbit. In particular, we show that the collection of perfect thin classes (a notion which is definable in the lattice of Π^0_1 classes) forms an orbit in the lattice of Π^0_1 classes; and a degree is anc iff it contains a perfect thin class. Hence the class of anc degrees is an invariant class for the lattice of Π^0_1 classes. We remark that the automorphism result is proven via a Δ^0_3 automorphism, and demonstrate that this complexity is necessary.

1. INTRODUCTION

While there are many ways of defining Π^0_1 classes, for the purposes of the present paper, we regard a (computably bounded) Π^0_1 class as the collection of (infinite) branches through an infinite binary tree. Alongside of computably enumerable sets and degrees, Π^0_1 class occupy a position as a fundamental notion in computability theory. In some sense, this is because they can be thought of as coding possible constructions. While this is very vague, a good illustration is provided by the fact that if I is a computable ideal in a computable ring R then the collection of prime ideals containing I forms a Π^0_1 class.

Π^0_1-classes have been studied for many years (for example, Jockusch and Soare [18], Kreisel [19]) and are particularly associated with models of Peano arithmetic (PA) and proof theoretical notions such as WKL_0. There are many ways of viewing Π^0_1 classes and many connections of computable mathematics with Π^0_1 classes. The main connection we have in mind is viewing Π^0_1-classes as being Stone spaces of logical theories, and in particular the Stone space associated with the lattice of c.e. filters $\mathcal{L}(\mathcal{Q})$, in $\mathcal{Q}$, a computable copy of the free Boolean algebra. Here, literals $\{p_i \mid i \in \omega\}$ can be viewed as propositions with $\neg p_i$ their negations, with $\wedge$ and $\vee$ having their usual interpretations. Then proper filters correspond to consistent theories,

Received by the editors March 12, 1999 and, in revised form, January 8, 2001.
2000 *Mathematics Subject Classification.* Primary 03D30; Secondary 03D25, 03D45.
The authors wish to thank the referee and Carl Jockusch for many very helpful comments which have greatly improved the readability and length of this paper. This research was supported by the Marsden Fund of New Zealand and the National Science Foundation.

computable filters to decidable theories, and ultrafilters to complete theories. We remind the reader that under the Stone duality, c.e. theories A correspond to the collection of complete theories $U(A)$ containing them; and conversely, to a Π^0_1-class C, viewed as the complete extensions of some theory, we can associate a theory $A(C) = T$, the intersection of the members of C. (While this is all standard, we will make some of these important connections more explicit in the next section.) For recent extensive surveys on Π^0_1 classes and their applications, we refer the reader to Cenzer [2], Cenzer-Jockusch [4], and Cenzer-Remmel [5].

The collection of Π^0_1 classes forms a lattice $\mathcal{L}(2^\omega)$. In this paper we study this lattice and its connection with the computably enumerable degrees, along the same lines as the well known Post program for the computably enumerable sets.

While there are many natural degree classes associated with a given Π^0_1 class $C \subseteq 2^\omega$, there is a natural way to associate a canonical degree with the class. The set of strings that have extensions in C is a co-computably enumerable set. Therefore we define $\deg_T(C)$ to be the degree of this co-c.e. set. Again we refer to section 2 for the reader who is unfamiliar with this material

The inspiration for the material of the present paper is the work of Soare and Martin, who demonstrated deep connections between definability and degree notions along the lines of Post's program. As is well known, Post sought a thinness property of the lattice of computably enumerable sets which guaranteed Turing incompleteness. In the deep paper [28], Soare demonstrated that this was impossible since all maximal sets were automorphic, and Martin [21] had earlier proved that the degrees containing maximal sets (indeed hyperhypersimple sets) were precisely the collection of all high degrees.

In the present paper we initiate a program similar to the above, but this time for the lattice $\mathcal{L}(2^\omega)$ of Π^0_1-classes under set inclusion. Our principal philosophy is that the study of $\mathcal{L}(2^\omega)$ can yield significant insight into computability in the same way that the study of $\mathcal{E}$ does.

The central concept of the present paper is that of a *thin* Π^0_1-class, which corresponds to a "maximal", or perhaps "hyperhypersimple", theory. Martin and Pour-El [22] constructed a perfect c.e. theory A (that is, essentially undecidable, or, viewed as a c.e. filter in the free Boolean algebra, the quotient of $\mathcal{Q}$ by A would be isomorphic to $\mathcal{Q}$) which was *maximal*. That is, A has the property that any c.e. theory A' containing A is a principal extension of A. In fact in the paper [22] Martin and Pour-El construct what seems a very special type of maximal theory. They construct what we now call a Martin–Pour-El theory. A Martin–Pour-El theory T is of the form $\langle p_i \mid i \in A, \neg p_j \mid j \in B\rangle$ where A and B are c.e. sets such that $|\mathbb{N} - (A \cup B)| = \infty$ and each extension of the theory T is principal over T.

It is easy to show that this type of theory is not definable in $\mathcal{L}(\mathcal{Q})$, since it must not only be maximal but additionally must be *well-generated*, that is, generated by literals and their negations (Theorem 5.8), but as we see in section 4 the notion of (perfect) maximal theory is definable.

Viewed via Stone duality, maximal theories correspond to *thin* classes. Here we say C is *thin* if it is infinite and for all Π^0_1 subclasses C' there is a clopen U such that $C' = C \cap U$.

What are the basic degree theoretical properties of thin classes?

In his thesis, Downey [9] proved that not every degree contains a Martin–Pour-El theory. He showed that while all high degrees contained Martin–Pour-El theories, and some low degrees, there were initial segments not containing them.

What was unusual was that there seemed something basic which prevented all c.e. degrees being realized, akin to the high permitting needed to ensure maximality, but somewhat different. To wit, the Martin–Pour-El *construction* was a rather unusual one since it had a certain "multiple permitting" character. In most arguments using the permitting technique, one has a series of "followers". For the j-th attack, one waits for an event to occur, and, when the event occurs ("realization"), one will begin the $(j+1)$-st attack on a bigger follower. If later any of the earlier followers gets permitted, then we win with suitable priority. A mild variation of this is that for R_e, we would need $g(e)$ many permissions for some function g, which is eventually constant for each attack.

For the Martin–Pour-El construction, each follower of the requirement R_e will need $f(\langle e, j\rangle)$-many attacks, and $f(n) > f(n-1)$, for some computable function f. That is, each follower needs more permissions than its predecessor for followers of the same requirement.

Eventually a new degree class, called the *anc* degrees, was introduced by Downey, Jockusch and Stob [12] to explain such arguments. It turns out that the anc degrees are a class containing all nonlow_2 degrees, and are closed upwards. They are exactly the degrees realized by many known constructions. We refer the reader to Downey, Jockusch and Stob [12]. However it is unknown if they are invariant for the computably enumerable sets, $\mathcal{E}$ or degrees $\mathbf{R}$.

Of interest to us is that Downey, Jockusch and Stob proved that each anc degree contains a maximal theory and in fact Martin–Pour-El's construction always yields a theory of anc degree. However, as mentioned above, the Martin–Pour-El construction needed the theory to be generated by literals or their negations, and this property is not definable.

Our first result is an analogue of Martin's theorem for $\mathcal{L}(2^\omega)$. We prove that if C is a perfect thin class then C has anc degree. Hence the anc degrees form an invariant class for $\mathcal{L}(2^\omega)$. [*This result says that the anc degrees occupy the position of the high degrees in the setting of* Π^0_1*-classes.*]

Naturally, having the analogue of Martin's theorem, we were interested in one to Soare's theorem. Could it be that any two thin classes are automorphic? To attack this question, our first problem was that there were no results for constructing automorphisms of $\mathcal{L}(2^\omega)$. Evidence from other structures said that the presence of additional algebraic structure can lead to quite different situations. For instance, Guichard [15] proved that the lattice of c.e. subspaces of an infinite dimensional fully effective vector space $L(V_\infty)$ has only countably many automorphisms, each induced by a computable semi-linear transformation of V_∞. In particular, if V_1 and V_2 are automorphic then they have the same 1-degree!

We show that any automorphism of $\mathcal{L}(\mathcal{Q})$ is induced by an automorphism of $\mathcal{Q}$, and further that if Φ is any automorphism of $\mathcal{L}(\mathcal{Q})$ induced by taking a c.e. set of generators to another, then in fact Φ is induced by a computable automorphism of $\mathcal{Q}$. Moreover, we can construct two thin Π^0_1-classes that are not automorphic via a Δ^0_2 automorphism. This seems to present an obstacle to our program.

Nevertheless, Remmel [24] has proved that $\mathcal{L}(2^\omega)$ has $2^{\aleph_0}$ automorphisms. We give a proof of Remmel's theorem in section 6. So there is some hope of an analogue of Soare's theorem. Indeed, as we show, this is the case. Using some new techniques, we are able to prove that if C_1 and C_2 are two perfect thin Π^0_1 classes, then C_1 is automorphic to C_2 via a Δ_3 automorphism. As we have seen above, this is the

sharpest result possible, since there are thin C_1 and C_2 such that there is no Δ^0_2 automorphism taking C_1 to C_2.

We see this paper as but a first step in an analysis of $\mathcal{L}(2^\omega)$ and its relationship with $\mathcal{E}$ and $\mathcal{C}$.

The plan of the paper is the following. Since we work in various settings that are connected by dualities of structure, we begin with a section detailing the correspondence between certain structures and the notation we will use in each setting.

In section 3 we consider the lattice $\mathcal{L}(2^{<\omega})$ of $2^{<\omega}$-c.e. filters, and develop some useful definitions for use in later sections.

Section 4 is concerned with thin Π^0_1-classes and their correspondence to Δ^0_2 Boolean algebras, and section 5 looks at perfect thin Π^0_1-classes.

Automorphisms of $\mathcal{L}(\mathcal{Q})$, the lattice of theories of $\mathcal{Q}$, are studied in section 6. We prove

Theorem 6.1. *Every automorphism of $\mathcal{L}(\mathcal{Q})$ is induced by a unique automorphism of $\mathcal{Q}$.*

Theorem 6.4. *Every automorphism of $\mathcal{L}(\mathcal{Q})$ induced by an isomorphism between two sets of c.e. generators of $\mathcal{Q}$ is computable.*

Theorem 6.2 (Remmel)**.** *There are $2^{\aleph_0}$ automorphisms of $\mathcal{L}(\mathcal{Q})$.*

However, we can also show

Theorem 6.6. *There are two thin Π^0_1-classes that are not Δ^0_2 automorphic.*

Despite this negative result, in section 7 we are able to prove analogues in $\mathcal{L}(\mathcal{Q})$ of Soare's and Martin's theorems for $\mathcal{E}$, the lattice of c.e. sets. We prove the theorems in the equivalent setting of Π^0_1-classes.

Theorem 7.8. *The anc degrees form an invariant class for the automorphism group of Π^0_1-classes.*

Theorem 7.9. *Any two perfect thin Π^0_1-classes are automorphic.*

2. Preliminaries

In this section we introduce our notation and some background results. It is a characteristic feature of the material that many of the results are proven much more easily in a particular setting. This is nothing new, and is the idea behind much of classical duality. We essentially work in three settings;

1. $\mathcal{Q}$, a computable copy of the free countable Boolean algebra, and its lattice of c.e. filters $\mathcal{L}(\mathcal{Q})$. We remind the reader that we view these as axiomatizable theories and the like as we describe below.
2. 2^ω, and in particular Π^0_1-classes, and its lattice of Π^0_1 classes $\mathcal{L}(2^\omega)$, and
3. $2^{<\omega}$, and in particular the lattice $\mathcal{L}(2^{<\omega})$ of c.e. filters on $2^{<\omega}$.

We write σ, τ for strings in $2^{<\omega}$, and λ denotes the empty string. For the length of σ we write $|\sigma|$, and if $|\sigma| = k$ then for $0 \leq i < k$ we write $\sigma(i)$ for the ith bit of σ. If τ is an extension of σ then we write $\sigma \prec \tau$, and $\sigma \preceq \tau$ denotes that either $\sigma \prec \tau$ or $\sigma = \tau$.

For $\sigma \in 2^{<\omega}$, $ext(\sigma) = \{\tau \mid \sigma \preceq \tau\}$ and, for nonempty σ, σ^- denotes the string of length $|\sigma| - 1$ contained in σ. If $x \in 2^\omega$ then we write $x \upharpoonright i$ to denote the member σ of $2^{<\omega}$ such that $\sigma = x(0)x(1)\dots x(i-1)$.

To distinguish between filters on $\mathcal{Q}$ and filters on $2^{<\omega}$ we will think of $\mathcal{Q}$ as the free Boolean algebra of propositional formulas modulo tautological equivalence. It then makes sense to refer to c.e. filters on $\mathcal{Q}$ as theories, and to ultrafilters on $\mathcal{Q}$ as complete theories. We regard $\mathcal{Q}$ as generated by $\{p_i \mid i \in \omega\}$, a computable set of free generators, with p_i and the negations $\neg p_i$ being referred to as literals.

Definition 2.1. Let T be a tree in $2^{<\omega}$ so that $T \subseteq 2^{<\omega}$ and T is closed under initial segments. Then $[T]$ will, as usual, be the set of all infinite branches $x \in 2^\omega$ such that $x \upharpoonright n \in T$ for all $n \in \omega$. A subset P of 2^ω is called a Π^0_1-class if there is a computable tree T in $2^{<\omega}$ such that $P = [T]$. Although the operation $[\cdot]$ is not 1-1, and hence has no "inverse", to a Π^0_1-class P, we can naturally associate a tree $P_< = \{\sigma \in 2^{<\omega} \mid \exists x \in P(\sigma \prec x)\}$. Note that $P = [P_<]$. (The reader should note that the definition of $P_<$ makes sense for any subset of 2^ω, and the equality $P = [P_<]$ holds provided that P is closed.)

Notice that while P is a set of infinite objects, $P_<$ is a countable set of strings, a Π^0_1 *set* rather than a Π^0_1 class. We remark that it is clearly a Π^0_1 set, since one can take any computable tree $T = \bigcup_s T_s$ representing P, and note that, by König's Lemma, for all $\sigma \in 2^{<\omega}$, $\sigma \in P_<$ if and only if

$$\forall s > |\sigma| \exists \tau \in T(|\tau| = s \wedge \sigma \prec \tau).$$

It is very useful to view Π^0_1 classes in $2^{<\omega}$ via their complements. The idea is that while there are many trees representing a particular class P, there is one tightest representation via the strings not in. For instance, let T_1 be the perfect tree above 1, and let T_2 be the tree consisting of the perfect tree above 1 together with a finite number of strings extending 0. Then $[T_1] = [T_2]$. Such considerations give rise to the following definition.

Definition 2.2. A subset G of $2^{<\omega}$ is called a $2^{<\omega}$-filter if

- $\sigma \in G \,\&\, \sigma \preceq \tau \implies \tau \in G$,
- $\sigma * 0 \in G \,\&\, \sigma * 1 \in G \implies \sigma \in G$.

Furthermore, $G \subseteq 2^{<\omega}$ is called a c.e. $2^{<\omega}$-filter if G is a c.e. set and is a $2^{<\omega}$-filter. For a (c.e.) subset G of $2^{<\omega}$, we let $\langle G \rangle$ denote the (c.e.) $2^{<\omega}$ filter containing G. (We also use the same notation in $\mathcal{L}(\mathcal{Q})$; that is, if G is a (c.e.) subset of Q then $\langle G \rangle$ denotes the (c.e.) $\mathcal{L}(\mathcal{Q})$ filter containing G.)

For a computably enumerable subset G of $2^{<\omega}$, $\langle G \rangle$ is c.e..

We have the following correspondence between Π^0_1-classes and computably enumerable filters. Its proof is straightforward, and is left to the reader.

Lemma 2.3. *A closed $P \subseteq 2^\omega$ is a Π^0_1-class if and only if $\overline{P_<}$ is a c.e. $2^{<\omega}$-filter.*

If G is a $2^{<\omega}$-c.e. filter then we may write $[\overline{G}]$ to denote the Π^0_1-class $\{x \in 2^\omega \mid \forall i\, (x \upharpoonright_{i+1} \notin G)\}$. We will tend to denote filters on $2^{<\omega}$ by F, G and H. We let $U(A)$ denote the set of complete theories containing A. We use T to denote a subset of $2^{<\omega}$. We use P and Q to denote Π^0_1-classes. We now elaborate on the dualities between the three settings.

Definition 2.4. Let $\mathcal{L}(2^{<\omega})$ denote the lattice of c.e. $2^{<\omega}$-filters ordered by set inclusion $\subseteq$ with $+$ and $\cap$ denoting least upper and greatest lower bounds respectively.

Note that $\mathcal{L}(2^{<\omega})$ is a distributive lattice with least element $\emptyset$ and greatest element $2^{<\omega}$.

Actually, as we now see, $\mathcal{L}(2^{<\omega})$ is really $\mathcal{L}(\mathcal{Q})$ in disguise. For the discussion below we use the notation $\epsilon_i p_i$ to denote one of p_i or $\neg p_i$. We have the following observation, which is surely known to anyone who has thought about it.

Lemma 2.5. *$\mathcal{L}(\mathcal{Q})$ and $\mathcal{L}(2^{<\omega})$ are computably isomorphic in a natural way.*

Proof. The theory $T \in \mathcal{L}(\mathcal{Q})$ corresponds to the c.e. filter $F_T = \{\sigma : T \vdash \sigma^*\}$, where σ^* is the conjunction of the p_i with $\sigma(i) = 1$ and the $\neg p_i$ with $\sigma(i) = 0$. One can check that the map induces an automorphism from $\mathcal{L}(\mathcal{Q})$ to $\mathcal{L}(2^{<\omega})$. □

We remark that, despite the "obvious correspondence" given in Lemma 2.5, as we will see, there are a number of conceptual advantages in sometimes thinking in terms of $\mathcal{L}(\mathcal{Q})$ and sometimes thinking of this as $\mathcal{L}(2^{<\omega})$.

Under the interpretation above, Π^0_1-classes correspond to complete theories in $\mathcal{Q}$ also in a very natural way. If x is a member of P then x is just an infinite binary path. We can then interpret this as a complete theory $A(x)$ equal to

$$\{\epsilon_i p_i \mid \epsilon_i p_i = p_i \text{ if } x(i) = 1, \text{ and } \epsilon_i p_i = \neg p_i \text{ if } x(i) = 0\}.$$

This is a complete theory on $\mathcal{Q}$, since precisely one of p_i or $\neg p_i$ will be in $A(x)$.

The point of Stone duality is that a logical theory B can be identified with the unique set $U(B)$ of all complete theories containing it, because $B = \bigcap\{D \mid D \in U(B)\}$. Since we are coding c.e. theories by Π^0_1 classes, we have a natural identification of theories B with Π^0_1 classes. Conversely, given a collection of complete theories E, one can always form the unique theory $T = \bigcap\{M \mid M \in E\}$. If E is a Π^0_1 class then T will be computably enumerable.

It is perhaps worthwhile to articulate further the manner by which this duality is obtained, since it is quite important for what is to follow. Think of a c.e. theory as being given in stages. Say, $B = \bigcup_s B_s$. Then we can build a natural representation of $U(B)$ in stages, via a computable tree T which is built inductively in stages. At stage 0, let $T_0 = \{\lambda\}$. At stage $s + 1$, we will have a tree T_s and will decide how to extend T_s to make T_{s+1}. For a string σ, we can interpret σ as $z(\sigma) = \bigvee_i \epsilon_i p_i$, where $\epsilon_i p_i = p_i$ if and only if $\sigma(i) = 1$. For ν on T_s, we simply put $\nu * j$ on T_{s+1} ($j \in \{0, 1\}$) iff $0 \notin \langle B_{s+1}, z(\nu * j)\rangle$ (the theory generated by B_{s+1} and $z(\nu * j)$ is not inconsistent). Conversely, given a Π^0_1 class P, with representing tree $T = \bigcup_s T_s$, one builds the theory B by putting $\bigvee \neg\epsilon_i p_i$ into B, for $\epsilon_i p_i = p_i$ iff $\sigma(i) = 1$, at the stage where there is no extension of σ in T_s.

3. The Lattice of $2^{<\omega}$-Filters

We now look at some key properties of this lattice. While, at least in terms of $\mathcal{L}(\mathcal{Q})$, some of these are well known, we will give a fairly detailed discussion for completeness.

Definition 3.1. Let $\mathcal{L}(2^{<\omega})(G, \uparrow)$ denote the sublattice

$$\{G' \in \mathcal{L}(2^{<\omega}) \mid G \subseteq G'\}.$$

Let $\mathcal{L}(2^{<\omega})(G, \downarrow)$ denote the sublattice

$$\{G' \in \mathcal{L}(2^{<\omega}) \mid G' \subseteq G\}.$$

(Similarly for theories.) We will drop the $2^{<\omega}$ and write, for instance, $\mathcal{L}(G, \downarrow)$, when the context is clear.

We say G is complemented in $\mathcal{L}(2^{<\omega})$ if there is some $\overline{G} \in \mathcal{L}(2^{<\omega})$ such that $G + \overline{G} = 2^{<\omega}$ and $G \cap \overline{G} = \emptyset$. In this case we will write $G \oplus \overline{G} = 2^{<\omega}$.

In $\mathcal{L}(\mathcal{Q})$ the interpretation of complementation is that a c.e. theory T has a complement T' if and only if $\langle T \cup T' \rangle = \mathcal{Q}$ and $T \cap T' = \{1\}$, where 1 here denotes the symbol for truth.

The dualities above allow us to characterize the complemented filters in $2^{<\omega}$. First we work in $\mathcal{Q}$. Notice that we have the correspondence, that a complemented filter in $2^{<\omega}$ corresponds to a Π^0_1 class which is complemented in 2^ω and hence to a complemented theory in $\mathcal{Q}$.

Lemma 3.2. *A c.e. theory T is complemented if and only if it is principal.*

Proof. $\Leftarrow$ is clear, since the filter theory generated by θ is complemented by that generated by $\neg\theta$. Conversely, suppose that A is not finitely generated and complemented by B. Thus the theory generated by A together with B is $\mathcal{Q}$, yet $A \cap B = \{1\}$. Since $0 \in \langle A \cup B \rangle$, for some $\theta \in A$ we have $\neg\theta \in B$. However, since A is infinitely generated, there is some $\varphi \in A$ with $\theta \nvdash \varphi$. Consequently, $1 \neq \neg\theta \vee \varphi \in A \cap B$. □

Corollary 3.3. *"T is principal" is definable in the lattice of c.e. theories.*

For the interpretation in $2^{<\omega}$, we use the following.

Definition 3.4. Suppose G is a c.e. $2^{<\omega}$-filter. Then we write $r(G)$ for the set of roots of G, that is,

$$r(G) = \{\sigma \in G \mid (\forall \tau)(\tau \in G \,\&\, \tau \preceq \sigma \implies \tau = \sigma)\}.$$

We have the following interpretation of Lemma 3.2.

Corollary 3.5. *An element G of $\mathcal{L}(2^{<\omega})$ is complemented if and only if $r(G)$ is finite.*

Proof. Under the isomorphism of Lemma 2.5, the roots of G form a finite set if and only if the associated theory is finitely generated, and hence principal. □

Reasoning classically about the Stone space of a logical theory, one would use the set of roots. However, in our case, we need to consider effective given objects. Therefore in place of the roots of G we use an effective generating set, which is the idea behind the following definition.

Definition 3.6. A *basis* of a $2^{<\omega}$-c.e. filter G is a subset B of G generating G and such that any two elements are $\preceq$ incompatible.

Note that $r(G)$ is a basis of G for any $2^{<\omega}$-filter G. $B \subseteq G$ is a basis of G iff B generates G and no proper subset of B generates G. Also G is complemented iff some basis of G is finite iff every basis of G is finite.

A useful fact is the following.

Lemma 3.7. *Let G be a $2^{<\omega}$-c.e. filter. Then there is a basis B of G which is c.e.*

Proof. Let $\{G_s\}$ be a computable enumeration of G. We construct a computable enumeration of a basis B of G as follows. Let $B_0 = \emptyset$. Given B_s, let n be the least natural number which exceeds the length of all the members of B_s, and then let $B_{s+1} = \{\tau \in 2^{<\omega} : |\tau| = n \wedge \tau \in G_s - \langle B_s \rangle\}$. It is easy to see that $B = \bigcup_s B_s$ is a basis of G. □

The proof above gives rise to the following reduction principle between elements of $\mathcal{L}(2^{<\omega})$.

Lemma 3.8. *For all $G_0, G_1 \in \mathcal{L}(2^{<\omega})$ there exist $G_0', G_1' \in \mathcal{L}(2^{<\omega})$ such that $G_0' \subseteq G_0$, $G_1' \subseteq G_1$, $G_0' \cap G_1' = \emptyset$ and $G_0 + G_1 = G_0' + G_1'$.*

Proof. Let $(\sigma_s^0)_{s\geq 0}$ and $(\sigma_s^1)_{s\geq 0}$ be computable enumerations of G_0 and G_1 respectively. We construct c.e. bases B_0 and B_1 of G_0' and G_1'. The construction is virtually identical to the one in Lemma 3.7 with the extra condition that elements of B_0 and B_1 are incomparable. (This is to ensure that if $\sigma \in G_0 \cap G_1 \neq \emptyset$ then $\sigma \notin G_0' \cap G_1'$.)

Then the $2^{<\omega}$-c.e. filters $G_i' = \langle B_i \rangle$ for $i = 0, 1$ witness the desired reduction. □

The lattice of c.e. subsets of an infinite c.e. set is always isomorphic to the lattice of c.e. sets. It is still an open question whether the analogous statement holds for $\mathcal{L}(G, \downarrow)$ for any G. It is pointed out in Cenzer-Jockusch [4], Theorem 6.3, that the Δ_3^0 version is false: there are $G_1, G_2 \in \mathcal{L}(2^{<\omega})$ such that there is no Δ_3^0 isomorphism taking $\mathcal{L}(G_1, \downarrow)$ to $\mathcal{L}(G_2, \downarrow)$. However, we do show that there are at most two isomorphism types.

Theorem 3.9. 1. *If $G \in \mathcal{L}(2^{<\omega})$ is nonempty and the root set $r(G)$ is finite, then $\mathcal{L}(2^{<\omega}) \cong \mathcal{L}(G, \downarrow)$.*
2. *If $G_0, G_1 \in \mathcal{L}(2^{<\omega})$ with both root sets $r(G_0)$ and $r(G_1)$ infinite, then $\mathcal{L}(G_0, \downarrow) \cong \mathcal{L}(G_1, \downarrow)$.*
3. *Furthermore, all the isomorphisms above are computable.*

Proof. Let G be a $2^{<\omega}$-c.e. filter.

1. Suppose $r(G) = \{\sigma_1, \ldots, \sigma_k\}$ for some $k \geq 1$.
   If $k = 1$ then let $\theta(\tau) = \sigma_1 * \tau$ for all $\tau \in 2^{<\omega}$.
   If $k > 1$ then for all $\tau \in 2^{<\omega}$ let $\theta(1^{k-1} * \tau) = \sigma_k * \tau$ and $\theta(1^i 0 * \tau) = \sigma_i * \tau$, for $i = 0, 1, \ldots, k-2$. Then θ generates the required isomorphism between $2^{<\omega}$ and $\mathcal{L}(G, \downarrow)$.
2. Suppose that $r(G)$ is infinite. Let G_m be the $2^{<\omega}$-c.e. filter
$$\{\sigma \in 2^{<\omega} \mid (\exists i < |\sigma|)(\sigma(i) = 1)\}.$$
   Observe that $r(G_m)$ is infinite and, in fact, $r(G_m) = \{0^j 1 \mid j \geq 0\}$. Let $(\sigma_s)_{s\geq 0}$ be a c.e. basis of G. Then the mapping
$$\theta : 0^s 1 * \tau \mapsto \sigma_s * \tau$$
   for $\tau \in 2^{<\omega}$ generates the isomorphism between $\mathcal{L}(G_m, \downarrow)$ and $\mathcal{L}(G, \downarrow)$.
3. It is clear that the isomorphisms above are computable.

□

Herrmann has conjectured that if $r(G)$ is infinite then $\mathcal{L}(G, \downarrow)$ is *not* isomorphic to $\mathcal{L}(2^{<\omega})$.

4. Thin Π_1^0-Classes and Boolean Algebras

The central notion for us in the rest of the paper is that of a thin Π_1^0-class. We like to think of Π_1^0-classes as subsets of the Cantor space 2^ω where the sets $I(\sigma) = \{x \mid \sigma \prec x\}$ form a basis of open intervals. Then any clopen subset of 2^ω is just a finite union of intervals.

Definition 4.1. 1. A Π_1^0 class P is called *thin* if P is infinite and for every Π_1^0 class $Q \subseteq P$ there is a clopen set $F \subset 2^\omega$ such that $Q = P \cap F$.

2. Suppose that T is a complete undecidable extension of T, but for all c.e. extensions T' of T, there is $\theta \in \mathcal{Q}$ such that $T' = \langle T, \theta \rangle$. Then, following Downey [9], we say T is a *maximal theory*.
3. Note that a Π^0_1 class is thin iff its corresponding theory is maximal.

Theorem 4.2. *"P is thin or finite" is definable in $\mathcal{L}(2^\omega)$.*

Proof. By Lemma 3.2, "T is principal" (and hence "C is clopen") is so definable. □

We remark that Open Question 6.3 in Cenzer-Jockusch [4] asks whether "T is finite" is definable in $\mathcal{L}(2^\omega)$.

For more results and background on thin Π^0_1-classes, see Cenzer, Downey, Jockusch and Shore [3] and Downey [9].

The following says that, in a sense, thin classes are the precise analogues of hyper-hyper-simple c.e. sets.

Lemma 4.3. *A nonempty Π^0_1-class P is thin if and only if $\mathcal{L}(2^\omega)(P, \downarrow)$ is an infinite Boolean algebra. In other words, P is thin if and only if the lattice of c.e. filters containing the associated theory $T(P)$ forms a Boolean algebra.*

Proof. To prove the "only if" part, assume that P is thin. Since P is infinite $\mathcal{L}(P)$ is infinite, and distributive, as we have already seen. (For any two distinct $x_1, x_2 \in P$, there is a clopen C with $x_1 \in P \cap C$ and $x_2 \notin P \cap C$.) Also, if $Q \subseteq P$ then Q has the form $C \cap P$ for some clopen C, so that $Q \cap (2^\omega - C)$ is the complement of Q in $\mathcal{L}(P)$.

Conversely, assume that $\mathcal{L}(P)$ is an infinite Boolean algebra. Then clearly P is infinite. Let Q be any Π^0_1 subclass of P. By assumption, Q is complemented in $\mathcal{L}(P)$. Let R be the Π^0_1 class such that $R \cup Q = P$ and $Q \cap R = \emptyset$. By the reduction principle, dualized from c.e. $2^{<\omega}$-filters, there exist Π^0_1 classes $Q' \supseteq Q$, $P' \supseteq P$ such that $Q' \cup R' = 2^\omega$ and $Q' \cap R' = Q \cap R = \emptyset$. Then Q' is clopen, and clearly $Q' \cap P = Q$. Thus P is thin. □

Lemma 4.3 is quite suggestive. For c.e. sets Lachlan characterized the lattice of supersets of hyperhypersimple sets as precisely the Σ^0_3 Boolean algebras. Since we now know that for maximal theories, the only supertheory lattices we get are Boolean algebras, perhaps, as in the c.e. set case, there is some characterization of the Boolean algebras that can be realized. Indeed this is the case.

Theorem 4.4. *The following classes of Boolean algebras coincide up to Δ^0_2 isomorphism.*

(i) $\{B \mid B$ *is an infinite* Δ^0_2 *Boolean algebra*$\}$.
(ii) $\{\mathcal{L}(T, \uparrow) \mid T$ *is a c.e. maximal theory*$\}$.
(iii) $\{\mathcal{L}(2^\omega)(P, \downarrow) \mid P$ *is a thin* Π^0_1*-class*$\}$.
(iv) $\{\mathcal{L}(2^{<\omega})(G, \uparrow) \mid [\overline{G}]$ *is a thin* Π^0_1*-class*$\}$.

Proof. We do (i) $\Leftrightarrow$ (iv), the others following by duality via Lemma 2.5. We first prove that (iv) $\subseteq$ (i). Suppose $\mathcal{L}(2^{<\omega})(G, \uparrow)$ is Boolean algebra, with P being the corresponding thin Π^0_1-class.

Let $\mathcal{B}$ be the Boolean algebra $(B, +, \cap, ^*)$, where

$$B = \{H \mid H \in \mathcal{L}(2^{<\omega})(G, \uparrow)\}$$

and $+, \cap$ and * are the operations on the lattice $\mathcal{L}(2^{<\omega})(G, \uparrow)$. We will give a Δ^0_2 presentation of $\mathcal{B}$.

Now we know that each extension of G is determined by a finite (root) set. The underlying set for $\mathcal{B}$ is the set of all finite subsets of $2^{<\omega}$, modulo the equivalence relation $\equiv$ where $F \equiv H$ iff $\langle G \cup F\rangle = \langle G \cup H\rangle$. Note that $\equiv$ is clearly c.e. The induced $+$, $\cap$ and complementation are obvious ones.

For the converse direction, let $\mathcal{B}$ be a Δ^0_2 Boolean algebra. By a result of Feiner [14] (see Downey [10], Corollary 3.10), we know that $\mathcal{B}$ is isomorphic to a c.e. presented Boolean algebra. Hence we can suppose, without loss of generality, that there is a c.e. theory F such that $\mathcal{B} \cong \mathcal{Q}/F$.

Remember here that, as in section 2, we think of $F = \bigcup_s F_s$ as being a set of elements of the form $\theta = \bigvee_i \epsilon_i p_i$. For this proof it is easiest to use the topological view of $\mathcal{Q}/F$ as a Π^0_1 class represented by a computable tree $T = \bigcup_s T_s$.[1]

We will define a Δ^0_2 map α from $2^{<\omega}$ to $2^{<\omega}$ and a Π^0_1 class $\widehat{T}$. For a node $\sigma(\theta)$ in T we will ensure that

- $\alpha(\sigma(\theta))$ exists iff $\sigma \in T$,
- α induces a homeomorphism from the Π^0_1 class $[T]$ to the Π^0_1 class $[\widehat{T}]$,
- $[\widehat{T}]$ is thin.

These three things will suffice for the theorem. (To see this, let $\widehat{F}$ represent $\widehat{T}$ in $\mathcal{Q}$. The point is that any extension of $\widehat{F}$ in $\mathcal{Q}$ will be finitely generated over $\widehat{F}$, as $\widehat{F}$ is thin. Hence the extensions of $\widehat{F}$ will correspond to the α-pre-images of strings $\widehat{\sigma}$ representing $\widehat{\theta} \notin \widehat{F}$. But these are just the elements θ of F, as α represents a homeomorphism.)

We will construct $\alpha = \lim_s \alpha_s$ in stages. At stage 0, we simply set $\alpha(\sigma) = \sigma$ for all $\sigma \in 2^{<\omega}$. We will need to meet the negative requirements

$$\mathcal{N}_\sigma : \sigma \in T \text{ iff } \lim_s \alpha_s(\sigma) \text{ exists} \in \widehat{T}.$$

Before we look at the precise nature of the satisfaction of the $\mathcal{N}_\sigma$, we look at the thinness requirements. Let $P_e \subset 2^{<\omega}$ be the e-th primitive recursive tree, so that $[P_0], [P_1], \ldots$ is an effective enumeration of all Π^0_1 classes.

We need to ensure that we meet the requirements

$$\mathcal{R}_e : [P_e] \subseteq [\widehat{T}] \to \exists C(C \text{ clopen } \& [P_e] = C \cap [\widehat{T}]).$$

Let W_e denote the set $2^{<\omega} - [P_e]$, the c.e. set of strings in the complement of $[P_e]$. The easiest way to understand the construction of a thin class is in the case that $[\widehat{T}]$ is perfect, which is covered by the case that $T = 2^{<\omega}$. Thus, here we will assume that $\sigma \in F$ for all σ, and hence we will need that $\lim_s \alpha_s(\sigma)$ exists for all $\sigma \in 2^{<\omega}$.

Now for all σ with $|\sigma| = e$ we regard $\mathcal{N}_\sigma$ as having higher priority than $\mathcal{R}_j$ for $j \geq e$.

The construction is similar to an e-state maximal set construction. The basic action is that we will try to define $\alpha(\sigma) \in W_{|\sigma|}$ if possible. Thus we will have a computable approximation $\alpha_s(\sigma)$ to $\alpha(\sigma)$.

At stage s we will have an approximation to $\widehat{T}_s$ of a certain height $h(s) \geq s$, and $[\widehat{T}]_s$ will be represented by those nodes ρ on $\widehat{T}_s$ such that $\exists \tau \in \widehat{T}_s(|\tau| = h(s) \wedge \rho \preceq \tau)$. $h(s)$ will be chosen so that $\alpha_s(\sigma)$ will be defined in $\widehat{T}_s$ for all $\sigma \in 2^s$. In fact, we simply ensure that 2^s equals $\alpha_s^{-1}([\widehat{T}]_s)$.

[1]Recall that $\theta = \bigvee_{i=1}^k \epsilon_i p_i$ is represented by a string $\sigma(\theta)$ as in Lemma 2.5. A string $\sigma(\theta)$ representing $\theta = \bigvee_i \epsilon_i p_i$ and all of its extensions dies at stage t if we see θ enter F_t.

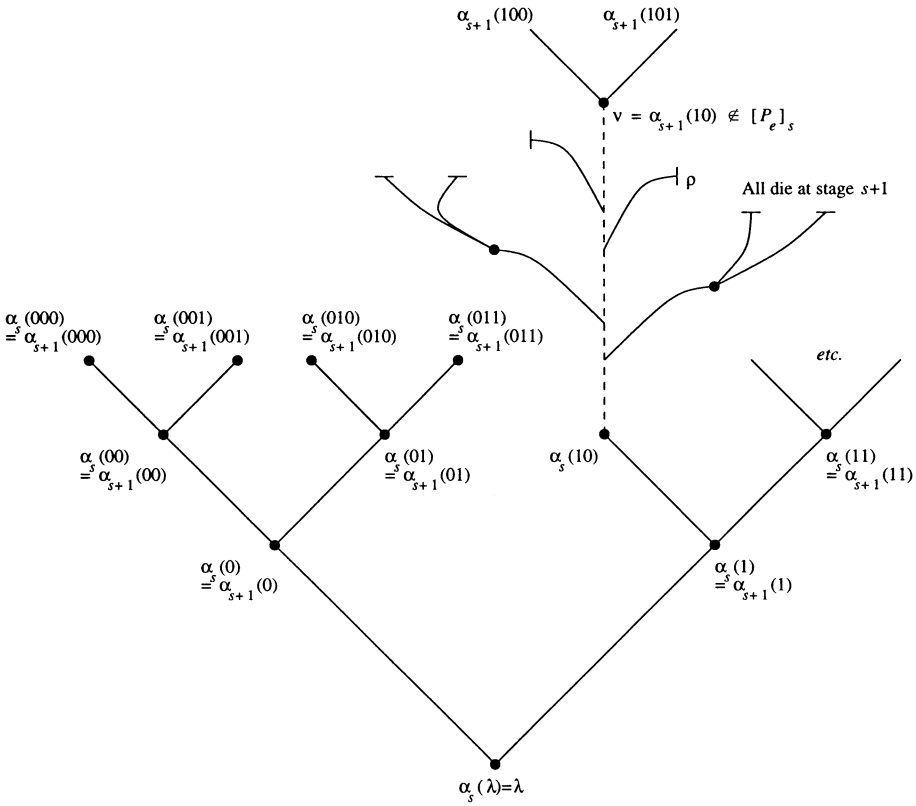

FIGURE 1. Example of the mapping α.

If no $\mathcal{R}_e$ acts at stage $s+1$ we simply extend α and its domain in the obvious way. That is, we set $h(s+1) = h(s)+1$, and for all strings σ of length $s+1$ we set $\alpha_{s+1}(\sigma * i) = \alpha_s(\sigma) * i$ for $i \in \{0,1\}$.

We act for the sake of $\mathcal{R}_{|\sigma|}$ if we see some $\nu \in [\widehat{T}]_s$ such that

- $\nu \in W_{|\sigma|^s}$ (since all its extensions in $P_{e,s}$ are killed by stage s),
- $\exists \sigma(|\sigma| = e+1 \wedge \alpha_s(\sigma) \prec \nu$ (so that ν is long enough to not injure any $\mathcal{N}_\gamma$ of higher priority),
- σ is not e-killed; that is, of the low e-state.

Note that we might as well take ν to have length s. The action is to redefine $\alpha_{s+1}(\sigma) = \nu$ (forcing $\alpha(\sigma) \in W_e$).

The details are then to extend the tree so that the partial isomorphism α_{s+1} is well defined. In detail, we then extend the tree so that $\widehat{T}_{s+1}$ pulls back to 2^{s+1}. That is, for all τ not extending $\alpha_s(\sigma)$ with τ having length s, let $\alpha_{s+1}(\tau * i) = \alpha_s(\tau) * i$, as above. Now for all β with $|\alpha_s(\sigma)| + |\beta| \leq s+1$, define $\alpha_{s+1}(\sigma * \beta) = \nu * \beta$. Declare σ as e-killed; having the high e-state. Initialize all $\mathcal{R}_j$ for $j > e$.

Finally, declare as terminal any ρ on $\widehat{T}_s$ extending $\alpha_s(\sigma)$ but incomparable with $\alpha_{s+1}(\sigma)$. See Figure 1, where $\sigma = 1 * 0$.

Notice that this action injures $\mathcal{N}_\tau$ for all τ with $\sigma \preceq \tau$, and $\mathcal{R}_j$ for $j > e$. The argument is finite injury. Since we e-kill σ when $\mathcal{R}_e$ receives attention via σ, one

can see that once $\mathcal{R}_e$ has priority, it can only further receive attention 2^{e+1} many times. Therefore, $\lim_s \alpha_s(\sigma)$ exists for all $\sigma \in 2^{<\omega}$, and the $\mathcal{N}_e$ are met. The $\mathcal{R}_e$ are met as follows. Go to a stage s_e such that for all $s \geq s_e$, $\mathcal{R}_e$ does not receive attention at s, and $\mathcal{R}_e$ has priority at stage s_e. Suppose that $[P_e] \subseteq [\widehat{T}]$. Then we see that for each σ of length $e+1$, the paths in $[\widehat{T}]$ above $\alpha(\sigma)$ are either disjoint from or equal to the paths in $[P_e]$ above $\alpha(\sigma)$. Hence $[P_e] = [\widehat{T}] \cap C$, where C is the clopen set determined by the finite collection of θ where the two classes are equal. This concludes the proof that there is a perfect thin class.

Now to complete the proof at hand all we need to do is to indicate the necessary modifications needed in the case that the domain of α is not all of $2^{<\omega}$ but is also in a state of formation. First we say that σ is *active* if σ has not yet been killed in T at stage s. That is, $\sigma \in [T]_s$. Our action at stage $s+1$ is that if σ becomes killed then we will immediately kill $\alpha_s(\sigma)$ and all its extensions in $[\widehat{T}]_s$, by declaring all its $h(s)$-long extensions as terminal. Finally, for $\mathcal{R}_e$ to receive attention, we replace

- σ is not currently e-killed

by

- σ is not currently e-killed *and σ is active.*

This concludes the proof of Theorem 4.4. □

Actually the proof above has a number of interesting corollaries. For instance, we can get a cheap proof of a result of Cenzer, Downey, Jockusch and Shore [3].

Corollary 4.5 (Cenzer et al. [3], Theorem 2.2)**.** (i) *For each computable ordinal β there is a countable thin Π^0_1 class $[\widehat{T}]$ of Cantor-Bendixson rank β.*

(ii) *Furthermore, if the rank is at least one, then we can take $\widehat{T}$ to be a computable tree with no dead ends.*

Proof. (i) follows from the above construction by choosing the template class $[T]$ to have rank β. It is very well known that there are countable Π^0_1 classes of arbitrary rank. Now for (ii). Note that there is something to prove here since even if we choose the template tree to have no dead ends, it does not mean that the thin class $[\widehat{T}]$ will have no dead ends. (ii) follows by modifying the construction as follows. When $\mathcal{R}_e$ receives attention at stage $s+1$, and we move $\alpha_{s+1}(\sigma)$ from $\alpha_s(\sigma)$ to ν, don't kill the strings ρ avoided, but just declare that they will have the trivial extension henceforth: we directly put $\rho * 0 * 0 * ...$ into $[\widehat{T}]$ and allow no other extension of ρ, with priority e. The argument still goes through. □

We also get the following corollary concerning the complexity of the lattices of Π^0_1 subclasses of a (thin) class.

Corollary 4.6. *Let P_e denote the e-th primitive computable tree. Then $\{\langle e, i\rangle : \mathcal{L}([P_e], \downarrow) \cong \mathcal{L}([P_i], \downarrow)\}$ is Σ^1_1-complete.*

Proof. It is well known that the pairs e, i such that the computable Boolean algebra $\mathcal{B}_e$ is isomorphic to the computable Boolean algebra $\mathcal{B}_i$ are Σ^1_1-complete. Since we know of no explicit proof of this in the literature, here is a proof. It is well-known (e.g. Rogers [25]) that the collection of paths through Kleene's $\mathcal{O}$ is Σ^1_1 complete. For each computable ordinal β, construct a canonical computable Boolean algebra as the interval algebra Intal(ω^β). These algebras are isomorphic iff the corresponding ordinals are the same. □

It would be interesting to know if the analog of the Slaman-Woodin ([27]) conjecture holds:

Open Question. Is $\{\langle e, i\rangle : P_e \text{ automorphic to } P_i\}$ Σ^1_1 complete?

The analogous result for the lattice of computably enumerable sets was proven by Cholak, Downey and Harrington [6].

5. Perfect Thin Classes

We now turn to perfect classes, our main concern. Recall that for a topological space X, the set X^d of derived points of X is the set of x such that $x \in cl(X - \{x\})$. If $x \in X - X^d$ then we say that x is an isolated point. A topological space is called *perfect* if it has no isolated points, that is, if $X = X^d$. In the Cantor space we have the following lemma, as noted in Cenzer, Downey, Jockusch and Shore [3].

Lemma 5.1. *For any thin Π^0_1 class P and any $x \in P$, x is computable if and only if x is isolated in P.*

Proof. Clearly, if $\{x\}$ is computable then $\{x\}$ is a Π^0_1 subclass of P. Since P is a thin Π^0_1 class, $\{x\}$ is a relatively clopen subclass of P and hence x is isolated.

The converse is trivial. □

Lemma 5.2. *P is a perfect thin Π^0_1-class if and only if $\mathcal{L}(2^\omega)(P, \downarrow)$ is an atomless Boolean algebra.*

Proof. Let P be a perfect thin Π^0_1 class. Then $\mathcal{L}(P, \downarrow)$ is a Boolean algebra by Lemma 4.3. Suppose $\mathcal{L}(P, \downarrow)$ is not atomless and let G be an atom. Then $G = \{x\}$ for some $x \in 2^\omega$, and it follows that $\{x\}$ is computable. Hence by Lemma 5.1, x is isolated in P, contradicting the assumption that P is perfect. Conversely, suppose that P is not perfect, but $\mathcal{L}(P, \downarrow)$ is an atomless Boolean algebra. Let x be an isolated point of P. Then x is computable and hence $\{x\}$ is an atom of $\mathcal{L}(P)$. □

Let x be an element of a Boolean algebra $\mathcal{B}$ and let $\mathcal{U}_x$ be the set of all ultrafilters on $\mathcal{B}$ containing x. Further, let $S(\mathcal{B})$ be the set $\{\mathcal{U}_x \mid x \in \mathcal{B}\}$. One version of the Stone representation theorem seen earlier is that every Boolean algebra $\mathcal{B}$ is isomorphic to a subset of $\mathcal{P}(S(\mathcal{B}))$, the power set Boolean algebra of $S(B)$. When we consider $S(B)$ as a topological space with basic open sets $\{\mathcal{U}_x \mid x \in \mathcal{B}\}$, called the Stone space, then we get the following characterization of Boolean algebras.

Theorem 5.3 (Stone)**.** *A Boolean algebra $\mathcal{B}$ is atomless if and only if the Stone space of $\mathcal{B}$ is perfect.*

Interpreting $\mathcal{L}(\mathcal{Q})$ as the lattice of c.e. logical theories, we can similarly say that a theory A is perfect if its corresponding Π^0_1-class is perfect.

We then have the interpretation of Lemma 5.3 in the setting of logical theories.

Theorem 5.4.
1. *A c.e. theory A is perfect if and only if A is consistent and is essentially undecidable.*
2. *"A is perfect" is definable in $\mathcal{L}(\mathcal{Q})$.*

For our purposes, the consequence of Theorem 5.4 we need is the following.

Corollary 5.5. *"A is a perfect maximal theory" is definable in $\mathcal{L}(\mathcal{Q})$.*

In section 7 we use the fact that being a perfect maximal theory is definable to obtain theorems for $\mathcal{L}(\mathcal{Q})$ that are analogous to Soare's and Martin's theorems for $\mathcal{E}$, the lattice of c.e. sets.

To finish this section we briefly discuss Martin-Pour-El theories, which were the first maximal theories to be constructed.

Martin–Pour-El theories are maximal perfect theories with a special set of generators. From Downey [9] we recall the following definitions in $\mathcal{L}(\mathcal{Q})$.

Definition 5.6. A c.e. theory T is *well-generated* if it is generated by a pair of sets $\{p_i \mid i \in A\}$ and $\{\neg p_i \mid i \in B\}$. (Here if T is consistent then $A \cap B = \emptyset$.)

Definition 5.7. A c.e. theory T is a Martin–Pour-El theory if it is well-generated and maximal.

Notice that if T is Martin–Pour-El then it is essentially undecidable, since $\omega - (A \cup B)$ is infinite. The reason that we concern ourselves with maximal theories rather than Martin–Pour-El ones is the following.

Theorem 5.8. *"T is Martin–Pour-El" is* not *definable in $\mathcal{L}(\mathcal{Q})$. In fact, there is an automorphism of $\mathcal{L}(\mathcal{Q})$ taking a Martin–Pour-El theory to a theory which is not Martin–Pour-El.*

This result is, of course, an immediate consequence of the main result of Section 7 (any two thin perfect Π^0_1 classes are automorphic; but a Martin–Pour-El theory corresponds to a thin perfect Π^0_1 class and there are thin perfect Π^0_1 classes which do not correspond to a Martin–Pour-El theory).

However, there is a very straightforward proof which runs as follows, and which we give for completeness. First we note the following. Let θ and ψ be any two nontrivial elements of $\mathcal{Q}$. Then there is a computable automorphism of $\mathcal{Q}$ sending θ to ψ. In particular, there is a computable automorphism Φ of $\mathcal{Q}$ taking p_1 to $p_1 \vee p_2$, where $M = \langle p_i : i \in A; \neg p_j : j \in B\rangle$ is Martin–Pour-El and $1 \in A$.[2] If the image of M were well-generated, then one of p_2 or p_1 would need to be in this image, and it is easy to argue that this cannot be the case.

6. Automorphisms of $\mathcal{L}(\mathcal{Q})$.

Take a computable copy of the free Boolean algebra $\mathcal{Q}$. Recall that $\mathcal{L}(\mathcal{Q})$ denotes the lattice of c.e. theories of $\mathcal{Q}$. In this section we study the automorphism group of $\mathcal{L}(\mathcal{Q})$.

Theorem 6.1. *Every automorphism of $\mathcal{L}(\mathcal{Q})$ is induced by a unique automorphism of $\mathcal{Q}$.*

Proof. Suppose $\Phi : \mathcal{L}(\mathcal{Q}) \mapsto \mathcal{L}(\mathcal{Q})$ is an automorphism. Let $p_0, p_1, \ldots$ be a countable set of generators for $\mathcal{Q}$. Since being a principal theory is definable in the

[2]To see this piece of folklore, the easiest way is to use yet another representation of $\mathcal{Q}$, namely the interval algebra representation. Recall that every computable Boolean algebra is computably isomorphic to the algebra of left closed right open subsets of a computable linear ordering. For the free algebra we can take $p_1 \mapsto [0, 1/2)$, $\neg p_1 \mapsto [1/2, 1)$, $p_2 \mapsto [0, 1/4) \cup [1/2, 3/4)$, etc. Rather than writing out all the painful details, we demonstrate the relevant isomorphism for our purposes. We show that $p_1 \mapsto p_1 \vee p_2$ is possible. So we send $[0, 1/2) \mapsto [0, 3/4)$. The linear map is defined piecewise as follows. For $x \in [0, 1/2)$ map x to $3/2 \cdot x$. For $x \in [1/2, 1)$ map x to $1 - \frac{1-x}{2}$. This will induce the desired automorphism of $\mathcal{Q}$. (The general case is essentially the same but has more pieces.)

language of theories of $\mathcal{Q}$, if A is a principal theory, then $\Phi(A)$ must also be a principal theory. Let $A_0 = \langle p_0 \rangle, A_1 = \langle p_1 \rangle, \ldots$, and so on.

Define $\phi(p_i) = \theta_i$, where $\Phi(A_i) = \langle \theta_i \rangle$. We now extend ϕ in the natural way to an automorphism of $\mathcal{Q}$ as follows. Let

$$\begin{aligned} \phi(p_i \vee p_j) &= \phi(p_i) \vee \phi(p_j), \\ \phi(p_i \wedge p_j) &= \phi(p_i) \wedge \phi(p_j), \\ \phi(\neg p_i) &= \neg \phi(p_i). \end{aligned}$$

For a formula θ, define $\phi(\theta)$ by induction from the above definitions.

Since "F is principal" is definable, and Φ is 1-1, we see that $\phi(a) = \phi(b)$ if and only if $a = b$. Now suppose $a \in \mathcal{Q}$. Then $\langle a \rangle$ is a principal theory. Since Φ is an automorphism, there is a $\theta \in \mathcal{Q}$ such that $\Phi(\langle \theta \rangle) = \langle a \rangle$. It follows that $\phi(\theta) = a$. Hence ϕ is 1-1 and onto.

To show ϕ is order-preserving, let $a, b \in \mathcal{Q}$. Then $a \leq b$ if and only if $\langle b \rangle \subseteq \langle a \rangle$ if and only if $\Phi(\langle b \rangle) \subseteq \Phi(\langle a \rangle)$ if and only if $\langle \theta_b \rangle \subseteq \langle \theta_a \rangle$, where $\langle \theta_b \rangle = \Phi(\langle b \rangle)$ and $\langle \theta_a \rangle = \Phi(\langle a \rangle)$. Furthermore, this occurs if and only if $\theta_a \leq \theta_b$. Since we must have $\phi(b) = \theta_b$ and $\phi(a) = \theta_a$, then $\phi(a) \leq \phi(b)$, as required.

Therefore $\phi : \mathcal{Q} \mapsto \mathcal{Q}$ is an automorphism.

We claim that ϕ induces Φ. That is, for $\widehat{W} = \{\phi(\theta) : \theta \in W\}$, we claim $\Phi(W) = \widehat{W}$. Otherwise there is some $\nu \notin \widehat{W}$ with $\nu \in \Phi(W)$; or some $\nu \in \widehat{W} - \Phi(W)$. Either case results in a contradiction because of the definability of principality and the definition of ϕ.

Finally we see that ϕ is unique. To see this, suppose ϕ_1 and ϕ_2 are automorphisms of $\mathcal{Q}$ inducing automorphisms of $\mathcal{L}(\mathcal{Q})$, Φ_1 and Φ_2 respectively. We must show that $\phi_1 \neq \phi_2$ implies $\Phi_1 \neq \Phi_2$. Suppose $\phi_1(a) \neq \phi_2(a)$. For a contradiction assume $\Phi_1(\langle a \rangle) = \Phi_2(\langle a \rangle) = \langle \theta \rangle$. Then $\langle \theta \rangle$ is a principal theory and $\phi_1(a) = \theta = \phi_2(a)$. Hence $\Phi_1(\langle a \rangle) \neq \Phi_2(\langle a \rangle)$. □

Theorem 6.2 (Remmel)**.** *There are $2^{\aleph_0}$ automorphisms of $\mathcal{L}(\mathcal{Q})$.*

Proof. Remmel proves this theorem from the interval algebra perspective of $\mathcal{Q}$, adapting ideas from Lachlan [20] to the Boolean setting. Here we will also use similar ideas to construct $2^{\aleph_0}$ automorphisms, but in the setting of $\mathcal{L}(2^{<\omega})$. Then from the duality we obtain $2^{\aleph_0}$ automorphisms between Π^0_1-classes, and hence $2^{\aleph_0}$ automorphisms of $\mathcal{L}(\mathcal{Q})$.

Let $F_0, F_1, F_2, \ldots$ be a computably enumerable listing of all c.e. $2^{<\omega}$-filters in $\mathcal{L}(2^{<\omega})$. We first construct a sequence of strings $(\sigma_s)_{s \geq 0}$ as follows. Recall that for $\sigma \in 2^{<\omega}$, $ext(\sigma) = \{\tau \mid \sigma \preceq \tau\}$ and, for nonempty σ, σ^- denotes the string of length $|\sigma| - 1$ contained in σ.

Let $\sigma_0 = \lambda$.

Suppose σ_n is defined, and define

$$\sigma_{n+1} = \begin{cases} \tau * 0 \text{ for } \tau = (\mu\nu)(\nu \in ext(\sigma_n) \cap F_n) & \text{if } ext(\sigma_n) \cap F_n \neq \emptyset, \\ \sigma_n * 0 & \text{if } ext(\sigma_n) \cap F_n = \emptyset. \end{cases}$$

Now let $f : \mathbb{N} \mapsto \{0, 1\}$. and define a mapping $\gamma_f(\nu)$ by strings, inducing an automorphism Φ_f, as follows:

1. Let $\gamma_f(\sigma_0 * \tau) = \sigma_0 * \tau$ for all τ such that $\sigma_1^- \not\preceq \sigma_0 * \tau$.

2. Suppose $\gamma_f(\sigma)$ is defined for all σ with $\sigma_n^- \not\prec \sigma$. We define $\gamma_f(\sigma)$ for all σ with $\sigma_n^- \prec \sigma$ and $\sigma_{n+1}^- \not\prec \sigma$. Suppose $\sigma = \sigma_n^- * i * \tau$. Then let

$$\gamma_f(\sigma) = \begin{cases} \gamma_f(\sigma_n^-) * i * \tau & \text{if } f(n) = 0, \\ \gamma_f(\sigma_n^-) * (1-i) * \tau & \text{if } f(n) = 1. \end{cases}$$

We then define $\Phi_f(F)$ to be $\{\gamma_f(\nu) : \nu \in F\}$.

Then, given a c.e. $2^{<\omega}$-filter F, we claim $\Phi_f(F)$ is also a c.e. $2^{<\omega}$-filter, and $\Phi_f^{-1}(F)$ exists and is a c.e. $2^{<\omega}$-filter. Observe that γ preserves $\prec$ and length. Suppose that $F = F_n$. First note that for all strings ν not extending σ_{n+1}, we can effectively calculate $\gamma_f(\nu)$, by knowing $\gamma_f(\sigma_j)$ for $j < n+1$. To demonstrate that $\Phi_f(F)$ is c.e., it thus suffices to argue for strings extending σ_{n+1}. There are two cases. Either there is no extension of σ_n in F_n, in which case there is nothing to prove, or there is some extension of σ_n in F_n. In the latter case, $\sigma_{n+1} = \tau * 0 \in F_n$, and hence the action of Φ_f is the identity on the filter $F_n \cap ext(\sigma_{n+1})$. The argument for Φ^{-1} is similar. For $F = F_n$, the inverse is determined by a finite number of computable partial maps given by the σ_i for $i \leq n$. Finally, if $f \neq g$ and n is the least number such that $f(n) \neq g(n)$, then $\Phi_f(ext(\sigma_n)) \neq \Phi_g(ext(\sigma_n))$.

Hence there are $2^{\aleph_0}$ automorphisms of $\mathcal{L}(2^{<\omega})$ and hence of $L(\mathcal{Q})$. □

We have seen that every automorphism of $\mathcal{L}(\mathcal{Q})$ is induced by an automorphism of $\mathcal{Q}$ and hence by its action on a generating set. One obvious approach to constructing automorphisms of $\mathcal{L}(\mathcal{Q})$ would be to induce such an automorphism as a permutation on $\{p_i \mid i \in \omega\}$, or some variation of this. The next two results demonstrate that *if* there are non-computable automorphisms of $\mathcal{L}(\mathcal{Q})$, they *cannot* be constructed along these lines. The ideas in the proofs to follow go back to Shore, first appearing in Kalantari's thesis.

Theorem 6.3. *Every automorphism induced by a permutation of literals is a computable automorphism.*

Proof. Let $\Phi : \mathcal{L}(\mathcal{Q}) \mapsto \mathcal{L}(\mathcal{Q})$ be an automorphism of $\mathcal{L}(\mathcal{Q})$ induced by the automorphism $\phi : \mathcal{Q} \mapsto \mathcal{Q}$, where ϕ is a permutation of literals $p_0, p_1, \ldots$.

We need the following 4 sets:

$$P_1 = \{p_{2i} \mid i \in \omega\},$$

$$P_2 = \{p_{2i+1} \mid i \in \omega\},$$

$$P_3 = \{p_{2i} \vee p_{2i+1} \mid i \in \omega\},$$

$$P_4 = \{p_{2i+1} \vee p_{2i+2} \mid i \in \omega\}.$$

Now since $p_j \mapsto p_{i_j}$ for some i_j, it follows that each P_i has an image set $\widehat{P}_i$. (This uses the fact that if $p_h \mapsto p_{i_h}$ and $p_k \mapsto p_{i_k}$ then $p_h \vee p_k \mapsto p_{i_h} \vee p_{i_k}$.)

Suppose we also know $\phi(p_0) = p_{i_0}$. We first show that ϕ is a computable permutation of literals. To find $\phi(p_1)$, find the unique member of $\widehat{P}_2$, p_{i_1} say, such that $p_{i_0} \vee p_{i_1} \in \widehat{P}_3$. Then $\phi(p_1) = p_{i_1}$. (The point here is that there is a unique p_j (namely p_1) in P_2 with $p_0 \vee p_1 \in P_3$, and hence, by the properties of automorphisms, and since the map is induced by a permutation of literals, the same must be true of the images.) Now repeat for p_2, looking for the unique $p_{i_2} \in \widehat{P}_1$ such that $p_{i_1} \vee p_{i_2} \in \widehat{P}_4$. Clearly ϕ is a computable permutation of the literals, and hence

is a computable automorphism of $\mathcal{L}$. Since ϕ induces Φ, we must have that Φ is a computable automorphism of $\mathcal{L}(\mathcal{Q})$. □

We remark that Theorem 6.3 only relies on certain independence properties of the p_i and Boolean combinations of the p_i, and has suitable generalizations, which say that every automorphism induced by a bijection between two sets of c.e. suitably independent generators of $\mathcal{Q}$ is a computable automorphism of $\mathcal{L}(\mathcal{Q})$.

One of our goals is to eventually prove that any two perfect thin classes are automorphic. The following result tells us that any such automorphism must be complicated. Recall that a function f *presents* an automorphism Φ if for all e we have $W_{f(e)} = \Phi(W_e)$, where here W_e denotes the eth c.e. theory.

It is an immediate corollary of the proof of Theorem 6.1 that the complexity of the presentation of an automorphism aligns itself with the complexity of the underlying automorphism of $\mathcal{Q}$.

Corollary 6.4. *Suppose that Φ is an automorphism of $\mathcal{L}(\mathcal{Q})$ presentable computably in $\mathbf{a}$. Then Φ is presentable by an automorphism of $\mathcal{Q}$ computable from $\mathbf{a}$, and conversely. In particular, every computable automorphism of $\mathcal{L}(\mathcal{Q})$ is induced by a computable automorphism of $\mathcal{Q}$.*

Proof. We note that if Φ is an automorphism of $\mathcal{L}(\mathcal{Q})$ then we can, in the presentation of Φ, determine its image on a set of generators $\{p_i \mid i \in \omega\}$ and hence determine $\phi(p_i)$ for the induced map ϕ, and hence $\phi(\theta)$ for all $\theta \in \mathcal{Q}$. From this one can always compute an index for the image of the set W under Φ as $\{\phi(\theta) : \theta \in W\}$. □

Most of the early constructions of automorphisms of the lattice of c.e. sets were effective in the sense that we could take f to be computable. Soare [28] revealed that this is not always the case by showing that all maximal sets were automorphic, yet there were maximal sets that were not automorphic by any Δ_2 automorphism. (That is, f could be chosen to be computable from the halting problem.) Despite the fact that our methods are very different, we can establish analogous results here for maximal theories, and, indeed, for Martin Pour-El theories.

Theorem 6.5. *There are two Martin–Pour-El theories that are not Δ^0_2 automorphic.*

Proof. We first recall how to construct a Martin–Pour-El theory

$$T = \langle \{p_i : i \in A\} \cup \{\neg p_j : j \in B\} \rangle$$

by sketching the proof from Downey [9]. For this proof, we let W_e denote the e-th c.e. theory, which is thought of as the e-th collection of formulas of the form $\bigvee_{i \in F} \epsilon_i p_i$, where, as usual, $\epsilon_i p_i$ is one of p_i or $\neg p_i$. We aim to meet the following set of requirements for all $e \in \mathbb{N}$:

$$\mathrm{R}_e : \exists \theta (\langle T, \theta \rangle = \langle T, W_e \rangle).$$

Let $d_0[s] < d_1[s] < d_2[s] < \dots$ be a list of the literals $\{p_i : i \notin A \cup B[s]\}$, ordered by least index.

To ensure our theory is perfect we meet the additional requirements

$$\mathrm{N}_e : \lim_s d_e[s] \text{ exists.}$$

To meet R_e we construct a finite set of formulas $Q_e = \lim_s Q_e[s]$ such that $x = \bigwedge Q_e$ is the witness for R_e. The argument is finite injury. The basic action is simple:

If we see some least $y \in W_e[s]$ such that $y \notin \langle T[s] \cup Q_e[s] \rangle$, then we say R_e requires attention via y at stage $s+1$. For the least e for which R_e requires attention we say R_e receives attention, and we define the set

$$L(e,y)[s+1] = \{\neg \epsilon_i d_i[s] \mid \epsilon_i d_i[s] \text{ occurs in } y \text{ and } i \geq e\}.$$

We then set $T[s+1] = \langle T[s] \cup L(e,y)[s+1] \rangle$ and let $Q_e[s+1] = Q_e[s] \cup \{y\}$. The result of this action is that if R requires attention via y at stage $s+1$ then $A[s+1] \vdash y \leftrightarrow z$, where z is a Boolean combination of $\{d_0[s], d_1[s], \dots d_{e-1}[s]\}$ (see Lemma 6.6 below). Since $\lim_s d_i[s] = d_i$ can be shown to exist for all i by induction, once R_e has priority and the limits have been reached for $i < e$, R_e can only require attention 2^{2^e} times because each time it requires attention a new Boolean combination of $\{d_0, d_1, \dots d_{e-1}\}$ is logically equivalent to y.

Lemma 6.6. *If R_e receives attention at stage $s+1$ via y, then there exists a Boolean combination z of*

$$\{d_0[s], d_1[s], \dots, d_{e-1}[s]\} = \{d_0[s+1], d_1[s+1], \dots, d_{e-1}[s+1]\}$$

such that $T[s+1] \vdash y \leftrightarrow z$.

Proof. Write y as a disjunction in the following way:

$$\bigvee_{i<e} \epsilon_i d_i[s] \vee \bigvee_{i \geq e} \epsilon_i d_i[s] \vee \bigvee_{\epsilon_i p_i \in T[s]} \epsilon_i p_i \vee \bigvee_{\neg \epsilon_i p_i \in T[s]} \epsilon_i p_i.$$

Thus y has the form $z \vee x \vee m \vee n$. Since $\vdash z \to y$, it suffices to show that $T[s+1] \vdash y \to z$. Now if $m \neq 0$ then $y \in T[s]$, since $\vdash m \to y$ and $m \in T[s]$. But then R_e does not require attention via y, and therefore we must have $m = 0$. Now $\neg n \in T[s+1]$ by definition of n and $\neg x \in T[s+1]$ by construction, and so we must have $T[s+1] \vdash y \to z$ as desired.

The goal is to construct two Martin–Pour-El theories T and $\widehat{T}$ which are not Δ^0_2 automorphic. For a c.e. theory H, consider set

$$H^* = \{e : W_e \subseteq H\}.$$

If T and $\widehat{T}$ are Δ^0_2 automorphic, then $T^* \leq_T \widehat{T}^* \oplus \emptyset'$. Thus it suffices to construct T and $\widehat{T}$ so that $\widehat{T}^* \leq \emptyset'$ yet $T^* \not\leq \emptyset'$. Let $\widehat{T}$ be any low Martin–Pour-El theory (Downey [9]). Then $\widehat{T}^* \leq_T \widehat{T}' \leq_T \emptyset'$. Thus it suffices to construct a T that is Martin–Pour-El and meets the requirements S_e below:

$$\mathrm{S}_e : \neg \forall i [T^*(i) = \lim_s \varphi_e(i,s)],$$

where φ_e denotes the e-th partial computable binary function. (In fact, we can suppose that $\varphi_e(i,s)$ is primitive computable.) For the requirement e the witness H_i is chosen by the recursion theorem, and we need to ensure that if the limit $\varphi_e(i) = \lim_s \varphi_e(i,s)$ exists, then it is different from $T^*(i)$. Hence we say that S_e requires attention at stage s, if

$$T^*(i)[s] = \varphi_e(i,s).$$

If S_e requires attention and $\varphi_e(i,s) = 0$, let $T_{s+1} = \langle T_s \cup H_i[s] \rangle$. If $\varphi_e(i,s) = 0$, choose a fresh $d_j[s]$ with $j \geq s$, and put $d_j[s]$ into $H_i[s+1]$ keeping it out of $T_{s'}$ for $s' \geq s$ with priority e, while $\varphi_e(i,s') = 1$. Note that once S_i receives attention j times, then we will never choose $d_j[s]$ as a witness to be put into H_i. Hence, in the same way, this will only injure the N_i finitely often, and is completely compatible with the Martin–Pour-El construction above. ∎

The proof above was suggested by the referee, and it replaced the original direct argument. Jockusch has asked whether a Martin–Pour-El theory of high degree can be Δ^0_2-automorphic to one of low degree.

7. An Invariant Class for $\mathrm{Aut}(\mathcal{L}(\mathcal{Q}))$

In this section we will give an analogue in $\mathcal{L}(\mathcal{Q})$ to Soare's and Martin's theorems for the lattice of c.e. sets $\mathcal{E}$, namely, an invariant class defined by an orbit of "maximal objects" in the automorphism group of $\mathcal{L}(\mathcal{Q})$. We first remind the reader of some definitions.

Definition 7.1. A *strong array* is a sequence of disjoint finite sets $\{F_n\}_{n\in\mathbb{N}}$ such that there is a computable function f with $F_n = D_{f(n)}$, where D_y denotes the finite set with canonical index y.

Further, a strong array is a *very strong array* if the following additional properties also hold:

1. $\bigcup_{n\in\mathbb{N}} F_n = \mathbb{N}$,
2. $0 < |F_n| < |F_{n+1}|$ for all $n \in \mathbb{N}$.

Definition 7.2. A c.e. set A is *array noncomputable (anc)* relative to a very strong array $\mathcal{F} = \{F_n\}_{n\in\mathbb{N}}$ iff

$$(\forall e)(\exists n)(W_e \cap F_n = A \cap F_n).$$

Then a c.e. degree $\mathbf{a}$ is *array noncomputable* (relative to $\mathcal{F}$) if there is a c.e. set $A \in \mathbf{a}$ such that A is array noncomputable (relative to $\mathcal{F}$).

Downey, Jockusch and Stob [12] showed that if $\mathcal{F}$ is a very strong array and $\mathbf{a}$ contains a c.e. set which is anc relative to *some* very strong array, then $\mathbf{a}$ also contains one which is anc relative to $\mathcal{F}$. That is, the array does not matter. In [12], those authors demonstrated that the anc degrees formed a class containing all non-low$_2$ c.e. degrees and some, but not all, low degrees, and that the anc degrees are closed upwards. The interest in anc degrees come from the fact that a number of constructions from the literature result in objects of exactly anc degrees. For instance, 4-tuples of c.e. sets A_1, A_2, B_1, B_2 with the property that every separating set of A_1 and A_2 is Turing incomparable with every separating set of B_1 and B_2 have the property that $A_1 \oplus A_2 \oplus B_1 \oplus B_2$ is of anc degree. Furthermore, if $\mathbf{a}$ is anc then $\mathbf{a}$ contains a 4-tuple of this form. A number of further results on anc degrees can be found in [12] and [13].

Of relevance to us here are the following results from Downey, Jockusch and Stob [12].

Theorem 7.3 (Downey, Jockusch and Stob)**.** *If M is a Martin–Pour-El theory, then M has anc degree.*

Theorem 7.4 (Downey, Jockusch and Stob)**.** *Each anc degree contains a Martin–Pour-El theory.*

Corollary 7.5. *Every anc degree contains a perfect maximal theory.*

Proof. Martin–Pour-El theories are maximal and perfect. □

As an analog with $\mathcal{E}$, an *invariant class* of c.e. degrees $\mathcal{C}$ is one where there is a set of c.e. theories $\widehat{\mathcal{C}}$ closed under automorphisms of $\mathcal{L}(\mathcal{Q})$ such that

$$\mathcal{C} = \{\deg(C) \mid C \in \widehat{\mathcal{C}}\}.$$

In the lattice of computably enumerable sets, Martin [21] established that the high c.e. degrees were invariant since the were precisely the degrees of the maximal (and hyperhypersimple) sets, which are definable in $\mathcal{E}$. We would like to demonstrate that the anc degrees are an invariant class for $\mathcal{L}(\mathcal{Q})$. The theorems above are quite suggestive of this via maximal theories. We have seen that being Martin–Pour-El is not invariant under automorphisms, but maximality is. Being a Martin–Pour-El theory depends upon being well-generated, but in the topological setting of perfect thin Π^0_1-classes we can dispense with the property of being well-generated and prove the following theorem, which improves Theorem 7.3 and shows that the anc degrees are the analog of the high degrees for $\mathcal{L}(\mathcal{Q})$.

Theorem 7.6. *Every perfect thin Π^0_1 class has anc degree.*

Proof. The proof is a kind of topological analogue of Theorem 4.9 of Downey, Jockusch and Stob [12]. The presentation of our proof runs parallel to their proof.

Let P be a perfect thin Π^0_1 class. Then $P = [T]$ for some computable tree T. In this and the next theorem, we find that splitting nodes play an important role.

Definition 7.7. Define $\mathrm{split}(P)$ to be the set

$$\{\sigma : \text{ Both } \sigma * 0 \text{ and } \sigma * 1 \text{ have extensions in } P\}$$

of nodes in $2^{<\omega}$. Since P is perfect, there is a unique isomorphism ϕ_P taking $(2^{<\omega}, \prec)$ to $(\mathrm{split}(P), \prec)$ with

$$\sigma_1 \leq_L \sigma_2 \quad \text{iff} \quad \phi_S(\sigma_1) \leq_L \phi_S(\sigma_2).$$

We order the nodes of $2^{<\omega}$ first by length, then by lexicographic order. This ordering induces a corresponding ordering of $(\mathrm{split}(P), \prec)$.

Since $\mathrm{split}(P)$ is a co-c.e. set of strings, we can only approximate the canonical map ϕ_P by a computable approximation $\lim_s \phi_{P,s}$, which we will denote by ϕ_s to save on notation.

(We want to be careful with our approximation of T, so that if $\phi_s(\nu)$ exists then $\phi_t(\nu)$ exists for all $t \geq s$. Choose a computable function h sufficiently large, that at stage s, if we define $\widehat{T}[0] = 2^{<\omega}$ and

$$\widehat{T}[s] = \{\sigma \in T : \exists \gamma \in T \ (|\gamma| = h(s) \wedge \sigma \preceq \gamma)\},$$

then $\widehat{T}[s]$ contains $\phi_s(\nu)$ for all $\nu \in 2^{<\omega}$ with $|\nu| \leq 2^{s+1}$. $\widehat{T}[s]$ is the approximation, at stage s, to the initial segments of members of P.)

Since the anc degrees are closed upwards, it is enough to construct a set A of anc degree with $A \leq_T P$. Let $\{F_n\}_{n\in\mathbb{N}}$ be a given very strong array. We suppose that $|F_n| = n + 1$. We have the following requirements for all $e \in \omega$:

$$\mathrm{R}_e : \exists n \, (W_e \cap F_n = A \cap F_n).$$

Here W_e denotes the e-th c.e. set. We reserve $F_{\langle e,0\rangle}, F_{\langle e,1\rangle}, \ldots$ for meeting requirement R_e and define the following computable function:

$$g(e, 0) = 2^{1+|F_{\langle e,1\rangle}|},$$
$$g(e, i) = 2^{g(e,i-1)+1+|F_{\langle e,i+1\rangle}|}.$$

To ensure that $A \leq_T P$ we insist that $x \in A[s + 1] - A[s]$ and $x \in F_{\langle e,i\rangle}$ implies that $\phi_s(\nu) \neq \phi_{s+1}(\nu)$ for some $\nu \in 2^{<\omega}$ with $|\nu| \leq g(e, i)$. Notice that this ensures that $A \leq_T P$, since we can generate the $\phi_s(\nu)$ from the extendible nodes of T. For

the construction below, and its verification, the reader should note that $g(e,i)$ is far in excess of $|W_e \cap F_{\langle e,i\rangle}|$.

The Construction. At stage 0, $A[0] = \emptyset$.

At stage $s+1$ for every e and i, if $W_e[s] \cap F_{\langle e,i\rangle} \neq A[s] \cap F_{\langle e,i\rangle}$ and $\phi_s(\nu) \neq \phi_{s+1}(\nu)$ for some ν with $|\nu| \leq g(e,i)$, enumerate all of $W_e[s+1] \cap F_{\langle e,i\rangle}$ into $A[s+1]$.

The Verification. We now demonstrate that every requirement R_e is satisfied. Suppose that R_e is the requirement with least e that is not satisfied. That means that $W_e \cap F_n \neq A \cap F_n$ for all n. We show how to construct a Π^0_1 class $Q \subseteq P$ which contradicts the thinness property of P. In fact we do this in the setting of $\mathcal{L}(2^{<\omega})$-c.e. filters and construct a nonprincipal extension V of the $2^{<\omega}$-filter M consisting of strings with no c.e. extension in P.

The splitting nodes play a big part. Let s_0 be a stage where $\phi_s(\sigma) = \phi(\sigma)$ for all strings σ with $|\sigma| = g(e,0)$. We define V from the parameter s_0. V is defined slowly, predicated on our failure to meet R_e.

Initially, we wait for a stage $s_1 > s_0$ where $W_e \cap F_{\langle e,0\rangle} \neq A \cap F_{\langle e,0\rangle}[s_1]$. At this stage put $\sigma(0,s_1) = \phi_{s_1}(0^{g(e,0)}) * 1$ into $V[s_1+1]$, and define a parameter $\tau(0,s_1) = 0^{g(e,0)}$. The reader should note that by the properties of s_0, σ_{0,s_1} is not a member of $\widehat{T}$.

We do nothing until a stage s_2 occurs where $W_e \cap F_{\langle e,1\rangle} \neq A \cap F_{\langle e,1\rangle}[s_2]$. At such a stage s_2 we put $\sigma(1,s_2) = \phi_{s_2}(0^{g(e,1)}) * 1$ into V, setting $\tau(1,s_2) = 0^{g(e,1)-g(e,0)}$.

Now at stage $t > s_2$, while $W_e \cap F_{\langle e,1\rangle} \neq A \cap F_{\langle e,1\rangle}[s_2]$, we treat s_2 as the same as s_0 and continue similarly for $g(e,2)$, etc. Note that if no changes occur to $\phi_s(\nu)$ for $|\nu| \leq g(e,1)$ after stage s_2, then additionally $\sigma(1,s_2) \notin \langle M \cup \{\sigma_{0,s_1}\}\rangle$. And we are well on the way to defining a non-finite extension of M.

The only problem is that perhaps we really do get a permission from $\widehat{T}$ below $\max_{|\nu|\leq g(e,1)} \phi_s(\nu)$. At the stage t where such a permission occurs, our action is to move right. That is, as given in the construction, we would correct $W_e \cap F_{\langle e,1\rangle} = A \cap F_{\langle e,1\rangle}[t]$, and put $\tau(0,t) = 0^{g(e,0)-1} * 1$. At the next stage t_2 where $W_e \cap F_{\langle e,1\rangle} = A \cap F_{\langle e,1\rangle}[t_2]$, we would use the string $\sigma(1,t_2) = \phi_{t_2}(\tau(0,t) * 0^{g(e,1)-g(e,0)}) * 1$ (which equals $\phi_{t_2}(\tau(0,t) * \tau(1,t)) * 1,$) in place of $\sigma(1,s_2)$.

More generally, we will have a set of strings $\tau(i,s)$ whose initial value is

$$0^{g(e,i)-g(e,i-1)}.$$

Each time permitting allows us to correct $A \cap F_{\langle e,i+1\rangle}$, we will move $\tau(e,i)$ one string to the right, amongst the strings ν of length $g(e,i)$, and additionally initialize $\tau(e,j)$ for $j > i$. At the next stage u that we have the inequality $W_e \cap F_{\langle e,i\rangle} \neq A \cap F_{\langle e,i\rangle}[u]$, we put $\sigma(i,u) = \phi_u(\tau(0,u) * \tau(1,u) * \cdots \tau(i,u) * 1)$ into V.

The key point is that for each i, $\lim_s \tau(i,s)$ and $\lim_s \sigma(i,s)$ both exist. This is because we only define $\sigma(i,s)$ in response to W_e on $F_{\langle e,i\rangle}$ and only change this in response to $\widehat{T}$ permission. Since the assumption is that we *fail* to meet R_e, we cannot get $\langle e,i\rangle$ permission, and hence will only need to redefine the τ and σ at most $\langle e,i\rangle - 1$ times. By construction, we see that the sequence $\sigma(i) = \lim_s \sigma(i,s)$ is a sequence independent over M, and hence V is not a principal extension of M.

So from part 2 of Theorems 4.2 and 5.4 (the definability of the properties of thinness and perfection), Theorem 7.6 and Corollary 7.5 above we have the following.

Theorem 7.8. *The anc degrees form an invariant class for the automorphism group of* $\mathcal{L}(\mathcal{Q})$.

This is the analogue of Martin's theorem for $\mathcal{E}$. We are now in a position to establish an analogue of Soare's theorem for $\mathcal{E}$.

Theorem 7.9. *Any two perfect thin Π^0_1 classes are automorphic.*

Proof. Let S and T be two perfect thin filters in $\mathcal{L}(2^{<\omega})$. Let $P_S = \{x \in 2^\omega : \forall n(x \upharpoonright n \notin S)\}$ denote S's associated Π^0_1 class, and similarly P_T. Let $\{\sigma_i : i \in \omega\}$ and $\{\tau_i : i \in \omega\}$ be two 1-1 computable enumerations of computably enumerable bases of S and T respectively, given via Lemma 3.7.

Any computable permutation p of ω induces an isomorphism from $\mathcal{L}(S, \downarrow)$ to $\mathcal{L}(T, \downarrow)$, as described below:

Let $F \in \mathcal{L}(S, \downarrow)$. Let B be a basis of F. Then $B = B_1 \cup B_2$ with B_1 and B_2 the disjoint sets described via

$$B_1 = \{\theta \in B : (\exists s)[\theta \prec \sigma_s]\},$$

$$B_2 = \{\theta \in B : (\exists s)[\sigma_s \preceq \theta]\}.$$

(Note that disjointness follows by transitivity of $\prec$ and the fact that B is a basis.)

Note that F is also generated by the basis $B' = B'_1 \cup B'_2$, where

$$B'_1 = \{\sigma_s : (\exists \theta \in B_1)[\theta \prec \sigma_s]\},$$

$$B'_2 = \{\sigma_s \nu : (\exists \theta \in B_2)[\sigma_s \preceq \theta \wedge \theta = \sigma_s \nu]\}.$$

Now we will map B to $\widehat{B}$, where $\widehat{B} = \widehat{B}_1 \cup \widehat{B}_2$, with

$$\widehat{B}_1 = \{\tau_{p(s)} : \sigma_s \in B'_1\} \quad \text{and} \quad \widehat{B}_2 = \{\tau_{p(s)}\nu : \sigma_s \nu \in B'_2\}.$$

The image of F is then the filter $\widehat{F}$ generated by $\widehat{B}$. Because p is a permutation, in a similar fashion, one can see that this map Γ is additionally onto, and hence is an isomorphism from $\mathcal{L}(S, \downarrow)$ to $\mathcal{L}(T, \downarrow)$. Since $\widehat{F}$ is found effectively from F, this is a computable isomorphism.

Note that the computable isomorphism above is induced by a computable map from the basis $\{\sigma_i : i \in \omega\}$ to $\{\tau_i : i \in \omega\}$ given by $\sigma_i \mapsto \tau_{p(i)}$. In the remainder of the proof, we will show how to define the computable permutation $p(i)$ so that the isomorphism can be extended to an automorphism of $\mathcal{L}(2^{<\omega})$.

From the previous theorem, the splitting nodes of P_S will be represented by $\phi_S(\nu)$ and similarly $\Phi_T(\nu)$ (Definition 7.7.) That is, $\text{split}(P_S) = \{\phi_S(\nu) : \nu \in 2^{<\omega}\}$. For $f \in 2^\omega$, let

$$\phi_S(f) = \bigcup\{\phi_S(\nu) : \nu \prec f\}.$$

Thus ϕ_S (and, similarly, ϕ_T) defines a natural homomorphism from 2^ω to S (resp. T).

The automorphism Φ of $\mathcal{L}(2^\omega)$ taking P_S to P_T is induced by a bijection $\Delta : 2^\omega \mapsto 2^\omega$. The map Δ is determined by the conditions that $\Delta(\phi_S(f)) = \phi_T(f)$ and $\Delta(\sigma_i f) = \tau_{p(i)} f$, where p is is a computable permutation of ω. These conditions are not in conflict since, for all f and i, $\sigma_i \not\prec \phi_S(f)$ and $\tau_i \not\prec \phi_T(f)$.

It is easy to see that Δ induces an automorphism of 2^ω taking P_S to P_T provided that

- for each Π^0_1 class P, $\Delta(P)$ and $\Delta^{-1}(P)$ are Π^0_1 classes.

We must define the permutation p to make this so.

The principal condition needed is that p is a computable permutation of ω with

$$(\forall \alpha \in 2^{<\omega})(\exists s_0)(\forall s \geq s_0)[\phi_S(\alpha) \preceq \sigma_s \text{ iff } \phi_T(\alpha) \preceq \tau_{(p(s))}].$$

Assume that p has the above property.

Lemma 7.10. *If Q is clopen, then $\Delta(Q)$ is clopen.*

Proof. It suffices to prove this when $Q = I(\sigma)$ for some string σ. We may assume, without loss of generality, that $\sigma = \sigma_i{}^\frown\sigma'$ for some i and σ', or $\sigma = \phi_S(\nu)$ for some ν. If $\sigma = \sigma_i{}^\frown\sigma$ then $\Delta(I(\sigma)) = I(\tau_{p(i)}{}^\frown\sigma)$. The second case is that $\sigma = \phi_S(\nu)$.

By the above property of p, for almost all s

$$\phi_S(\nu) \preceq \sigma_s \leftrightarrow \phi_T(\nu) \preceq \tau_{p(s)}.$$

There is a finite collection H of indices such that for all $i \in H$, $\phi_S(\nu) \not\preceq \sigma_{p^{-1}(i)}$, yet $\phi_T(\nu) \preceq \tau_i$, and similarly a finite collection J such that for all $j \in J$, $\phi_S(\nu) \preceq \sigma_j$ yet $\phi_T(\nu) \not\preceq \tau_{p(j)}$. This allows us to define

$$\Delta(I(\phi_S(\nu)) = \left(I(\phi_T(\nu)) \cup \left(\bigcup_{j \in J} I(\tau_{p(j)}) \right) \right) \cap \left(\bigcap_{i \in H} \overline{I(\tau_i)} \right).$$

□

Lemma 7.11. *If P is a Π^0_1 class, then so are $\Delta(P)$ and $\Delta^{-1}(P)$.*

Proof. As we have seen above, for any $F_0 \subseteq P_S$, the mapping $F_0 \mapsto \langle \{\tau_{p(i)}\nu : \sigma_i\nu \in F_0\}\rangle$ induced by Δ is a computable isomorphism from $\mathcal{L}(S, \downarrow)$ to $\mathcal{L}(T, \downarrow)$, and hence from $\mathcal{L}(P_S, \uparrow)$ to $\mathcal{L}(P_T, \uparrow)$.

We claim that any $F \in \mathcal{L}(2^{<\omega})$ is of the form

$$\langle \{\phi_S(\nu) : \nu \in H\} \cup F_0\rangle$$

for some *finite* set H, and $F_0 \in \mathcal{L}(2^{<\omega})$, with $F_0 \subseteq S$. To see this, let $F_0 = F \cap S$. To obtain H, note that $P_S \cap P_F = P_S \cap Q$ for some clopen set Q, as P_S is thin. Note also that $P_S = [T]$, where T is the (noncomputable) tree of all strings σ with $\sigma \preceq \phi_S(\nu)$ for some $\nu \in 2^{<\omega}$. The set $T - Q$ can be expressed as a finite union of intervals $I(\sigma)$ for $\sigma \in T$, and hence $\bigcup_{\nu \in H_0} I(\phi_S(\nu))$, for some finite $H_0 \subset 2^{<\omega}$. By König's Lemma, if $\nu \in H_0$, then for all sufficiently large τ with $\nu \preceq \tau$ we have $\phi_S(\tau) \in F$. Choose n so large that it exceeds the length of all strings in H_0, and $\phi_S(\sigma) \in F$ for all strings σ of length n which extend any string in H_0. Let $H = \{\nu' : (\exists \nu)[\nu \in H_0 \wedge \nu \preceq \nu' \wedge |\nu'| = n]\}$. Then this H works, since $\nu \in H$ implies $I(\nu) \cap P_S \subseteq (2^\omega - P_F)$.

Let $P = P_F$ be a Π^0_1 class. As above, we have F of the form $\langle \{\phi_S(\nu) : \nu \in H\} \cup F_0\rangle$, and hence P of the form $Q \cap P_0$ for some clopen Q and Π^0_1 subclass P_0 of P. Then $\Delta(P) = \Delta(Q) \cap \Delta(P_0)$. But we have proven that $\Delta(Q)$ is a Π^0_1 class since Q is clopen, and $\Delta(P_0)$ is a Π^0_1 class as p is computable. Hence $\Delta(P)$ is a Π^0_1 class. The proof for Δ^{-1} is essentially symmetrical.

That almost concludes the proof of the theorem. It remains to prove that there is a permutation p which is computable and satisfies the hypothesis above.

We have already noted that split(G) is a Π^0_1 set for $G \in \{S, T\}$. Let split$_s(G)$ be strings which appear to be splits on G at stage s. Define $\phi_{s,S}(\nu)$ be the unique computable isomorphism taking $(2^{<\omega}, \prec)$ to $(\text{split}_s(S), \prec)$ with

$$\sigma_1 \leq_L \sigma_2 \quad \text{iff} \quad \phi_{s,S}(\sigma_1) \leq_L \phi_{s,S}(\sigma_2),$$

(and similarly there is a unique $\phi_{s,T}$). Then $\lim_s \phi_{s,S}(\alpha) = \phi_S(\alpha)$ (and similarly for T).

Now we will define p stagewise using $\phi_{s,S}$ and $\phi_{s,T}$. At an even stage e, find the least $i \leq e$ such that σ_i is not in the domain of p_{e-1} (if no such i exists, do nothing). Since $\lim_s \phi_{s,S}(\alpha) = \phi_S(\alpha)$, $\lim_s \phi_{s,T}(\alpha) = \phi_T(\alpha)$ and $\{s : \phi_T(\alpha) \prec \tau_s\}$ is infinite (otherwise T is not thin), there exist t and j such that τ_j is not in the range of p_{e-1} and, for all α, $\phi_{t,S}(\alpha) \preceq \sigma_i$ iff $\phi_{t,T}(\alpha) \preceq \tau_j$. Let $p(i) = j$. At odd stages we will take similar action ensuring p is onto. It is easy to see this meets the hypothesis of p. □

Corollary 7.12. *Any two perfect thin Π^0_1 classes are Δ_3-automorphic.*

Proof. It is enough to show that the complexity of the automorphism we constructed in the above proof is Δ_3. In particular, our goal is to find a Δ_3 function f such that if W_e is a c.e. filter then $\Phi(W_e) = W_{f(e)}$. The value of $\Phi(W_e)$ depends on whether $W_e \subseteq S$ or not. Determining which of these cases holds is Δ_3. (Inclusion for c.e. sets is a complete Π_2 relation. So inclusion for c.e. filters is a complete Π_2 relation.) If $W_e \subseteq S$ then, as we noted above, $\Phi(W_e)$ can be found effectively from the map p. Otherwise, by carefully examining the last two lemmas, we can see that $\Phi(W_e)$ can be found effectively in $\mathbf{0}''$. □

8. Remarks

One can look more generally at automorphisms of the classes. As we saw in Theorem 4.4, for any Δ^0_2 Boolean algebra $\mathcal{B}$, there is a theory F such that $\mathcal{Q}/F \cong \mathcal{B}$, where F corresponds to a thin Π^1_0-class. We have see that if $\mathcal{B}$ is an atomless Boolean algebra, then this is enough to guarantee an orbit. Are there any other such $\mathcal{B}$?

The strongest theorem would be that if $\mathcal{B}_1 \cong \mathcal{B}_2$ then F_1 is automorphic to F_2 (where $\mathcal{B}_i$ corresponds to F_i as above). This would require significant technology, since there are computable Boolean algebras that are not even arithmetically isomorphic. This is even true of rank 1 Boolean algebras (Downey-Jockusch [11]). A good test case is to consider whether, when $\mathcal{B}_1$ and $\mathcal{B}_2$ are computable copies of the Boolean algebra of finite and cofinite sets, are $\mathcal{B}_1$ and $\mathcal{B}_2$ automorphic?[3]

The proof technique from Theorem 7.6 is enough to establish the following:

Theorem 8.1. *Suppose that S and T are thin $2^{<\omega}$ c.e. filters with $\mathcal{L}(S, \uparrow)$ isomorphic to $\mathcal{L}(T, \uparrow)$. Let R be a computable tree representing the Δ^0_2 Boolean algebra isomorphic to $\mathcal{L}(S, \uparrow)$. Let ϕ_S be any isomorphism from $\mathrm{split}(P_R)$ to $\mathrm{split}(P_S)$, and define ϕ_T similarly. Again let σ_i and τ_i denote bases for S and T, respectively. Suppose that p is any permutation such that the map $\sigma_i \mapsto \tau_{p(i)}$ induces an isomorphism from $\mathcal{L}(S, \downarrow)$ to $\mathcal{L}(T, \downarrow)$.*

If p also satisfies

$$(\forall \alpha \in 2^{<\omega})(\exists s_0)(\forall s \geq s_0)[\phi_S(\alpha) \preceq \sigma_s \text{ iff } \phi_T(\alpha) \preceq \tau_{(p(s))}],$$

then S and T are automorphic.

Another area of interest is the lattice of subfilters of a fixed ultrafilter. Here, $=^*$ is a congruence. We already know this is a rich object, since we can effectively embed $\mathcal{E}^*$ here (Downey [8, 9]).

[3]Since the first draft of this paper, Cenzer and Remmel have obtained some results about this case. See [4].

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